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ON THE SECOND THEOREM OF THE MEAN*

BY CHARLES N. HASKINS

1. Introduction. The proofs of the second theorem of the mean are, with one exception, based either on the lemma of Abel or on the process of integration by parts.† The exception is the proof of Netto‡ which, however, imposes unnecessary restrictions on the functions involved, and is, moreover, even under these restrictions, incomplete.§ In the present note the lacuna of Netto's proof is filled, and, by use of the fundamental theorem of Fourier's constants,‖ the restrictions on the vanishing of the functions involved are removed.

2. The Second Theorem of the Mean. In the interval $a \leqq x \leqq b$ let the function $f(x)$ be single-valued, bounded, and monotonic; and let the function $\phi(x)$ be single-valued, and, as well as $[\phi(x)]^2$, integrable.¶ Then the second theorem of the mean asserts that

$$(1) \qquad \int_a^b f(x)\, \phi(x)\, dx = f(a+) \int_a^\xi \phi(x)\, dx + f(b-) \int_\xi^b \phi(x)\, dx, \qquad a \leqq \xi \leqq b.$$

It is worth noting that if $\phi(x) \geqq 0$ the second theorem of the mean is an obvious consequence of the maximum-minimum theorem, provided $f(a+)$ and $f(b-)$ are the limits of indetermination of $f(x)$ in (a, b), in which case, moreover, $f(x)$ is not restricted to being monotonic. The two theorems are in a certain sense complementary; in the maximum-minimum theorem the sign of $\phi(x)$ is restricted, but $f(x)$ need not be monotonic; in the second theorem of the mean the restriction on the sign of $\phi(x)$ is removed, but, as a compensation, the function $f(x)$ must be monotonic, as is shown by examples in which the theorem fails when $f(x)$ is not monotonic.*

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* Presented to the American Mathematical Society, New York, Feb. 28, 1908.

† Cf. Pringsheim, *Münchener Sitzungsberichte*, vol. 30 (1900), pp. 209–233.

‡ Netto, *Zeitschrift für Mathematik und Physik*, vol. 40 (1895), p. 180.

§ Pringsheim, loc. cit., p. 232.

‖ Cf. de la Vallée-Poussin, *Annales de la société scientifique de Bruxelles*, vol. 17 (1892–3), p. 18. Hurwitz, *Mathematische Annalen*, vol. 57 (1903), p. 425.

¶ The integrability of $[\phi(x)]^2$ brings about also the integrability of $|\phi(x)|$. The theorem is true even when $|\phi(x)|$ is not integrable provided $f(x) \cdot \phi(x)$ is so. The condition that $[\phi(x)]^2$ be integrable is introduced in order that the fundamental theorem of Fourier's constants may be used.

3. Netto's proof and its limitations. If we discard the trivial case in which $f(a+) = f(b-)$, the relation (1) can be reduced to the simpler form

$$(2) \qquad \int_a^b F(x)\,\phi(x)\,dx = \int_a^{\xi} \phi(x)\,dx, \qquad a \leqq \xi \leqq b;$$

where

$$F(x) \equiv \frac{f(b-) - f(x)}{f(b-) - f(a+)}, \qquad a < x < b,$$

$$F(a) = F(a+) = 1, \quad F(b) = F(b-) = 0.$$

That is, if there exist a ξ for which either (1) or (2) is true then for that same ξ the other is true.

Netto's proof now proceeds as follows:

$$\text{let} \qquad K(x) \equiv \int_a^x F(x)\,\phi(x)\,dx, \qquad L(x) \equiv \int_a^x \phi(x)\,dx;$$

and consider the roots $a_1, a_2, \cdots$ of the equation

$$\phi(x) = 0.$$

It follows from the first law of the mean that the relation (2) holds in the interval $a \leqq x \leqq a_1$, and Netto then proves that if the relation holds in any interval $a \leqq x \leqq A$ it is possible to assign a $\delta > 0$ such that the relation holds in the interval $a \leqq x \leqq A + \delta$. From this he concludes that the relation must hold throughout the entire interval $a \leqq x \leqq b$. Now, as Pringsheim† points out, this conclusion involves the restriction that $\phi(x) = 0$ shall have a *first* root $a_1 > a$, and the assumption that the points $A + \delta$ have not a limiting point $B < b$.

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* Cf. Kronecker-Netto, *Vorlesungen über die Theorie der einfachen und vielfachen Integrale*, p. 63. Pierpont, *Theory of Functions of Real Variables*, vol. 1, p. 380.

These examples leave open the question as to whether the validity of the theorem does not depend on the fact that $f(a+)$ and $f(b-)$ are the limits of indetermination of $f(x)$, rather than on the fact that $f(x)$ is monotonic. A very simple modification of Kronecker's example shows, however, that the latter is the determining condition.

† Pringsheim, loc. cit., p. 232.

4. Proof of the non-existence of a limiting point. Under the hypotheses concerning $f(x)$ and $\phi(x)$ the two integrals $K(x)$ and $L(x)$ are continuous functions of x in the interval $a \leqq x \leqq b$. The assertion that for every x in the interval $a \leqq x \leqq A$ there exists a ξ, such that

$$K(x) = L(\xi), \qquad a \leqq \xi \leqq x,$$

is therefore equivalent to the assertion that in any interval $a \leqq x \leqq \bar{x} \leqq A$ the minimum and the maximum values of the continuous function $K(x)$ are respectively not less than the minimum and not greater than the maximum value of the continuous function $L(x)$. In other words,

$$\min \{L(x)\} \leqq \min \{K(x)\} \leqq \max \{K(x)\} \leqq \max \{L(x)\}, \qquad a \leqq x \leqq \bar{x} \leqq A.$$

If, then, there exist a limiting point B such that if $a \leqq x < B$, there always exists a ξ for which

$$K(x) = L(\xi), \qquad a \leqq \xi \leqq x < B;$$

but such that there exists no β for which

$$K(B) = L(\beta), \qquad a \leqq \beta \leqq B;$$

it follows that either

$$\left.\begin{array}{l} K(B) < \min L(x) \\ K(B) > \max L(x) \end{array}\right\}, \qquad a \leqq x \leqq B.$$

or

This, however, is impossible. For if $K(B)$ is neither the maximum nor the minimum value of $K(x)$ in $a \leqq x \leqq B$ then there exists an $x^* < B$ such that $K(x^*) = K(B)$, and corresponding to this x^*, a ξ^* such that

$$K(x^*) = L(\xi^*), \qquad a \leqq \xi^* \leqq x^* < B.$$

Hence

$$K(B) = L(\xi^*), \qquad a \leqq \xi^* < B.$$

Assume, then, that $K(B) = \max \{K(x)\}$, $a \leqq x \leqq B$. Take a set of values $x_1 < x_2 < x_2 \cdots < B$ approaching B as a limit. By the hypothesis, to each x_i corresponds at least one ξ_i such that

$$K(x_i) = L(\xi_i), \qquad a \leqq \xi_i \leqq x_i < B.$$

From the continuity of $K(x)$ follows that

$$\lim_{i=\infty} L(\xi_i) = \lim_{i=\infty} K(x_i) = K(B).$$

Now either

$$\left.\begin{array}{lll} & (1) & \min \{L(x)\} \leqq K(B) \leqq \max \{L(x)\} \\ \text{or} & (2) & K(B) < \min \{L(x)\} \\ \text{or} & (3) & \max \{L(x)\} < K(B) \end{array}\right\}, \quad a \leqq x \leqq B.$$

But (2) and (3) are impossible. For if (2) were true we could find an x_i with corresponding ξ_i such that

$$L(\xi_i) = K(x_i) < K(B) < \min \{L(x)\}, \quad a \leqq \xi_i \leqq x_i < B;$$

and if (3) were true we could find an x_i with corresponding ξ_i such that

$$\max \{L(x)\} < K(x_i) = L(\xi_i) < K(B), \quad a \leqq \xi_i \leqq x_i < B;$$

both of which are impossible.

Hence
$$\min \{L(x)\} \leqq K(B) \leqq \max \{L(x)\}, \qquad a \leqq x \leqq B,$$

and hence there exists a β for which

$$K(B) = L(\beta), \qquad a \leqq \beta \leqq B.$$

The case $K(B) = \min \{K(x)\}$, $a \leqq x \leqq B$, may be treated in the same way and with the same result. Hence:

There exists no limiting point B such that for all $x < B$ a ξ can be found for which

$$K(x) = L(\xi), \qquad a \leqq \xi \leqq x < B,$$

but such that no ξ can be found for which

$$K(B) = L(\xi), \qquad a \leqq \xi \leqq B.$$

5. Application of the fundamental theorem of Fourier's constants. For convenience we will now take $a = 0$, $b = 2\pi$, which may always be done by a linear transformation of the variable x. The assumptions concerning the nature of $f(x)$ and $\phi(x)$ ensure the existence of the Fourier's constants

$$a_k = \frac{1}{\pi}\int_0^{2\pi} F(x)\cos kx\,dx, \qquad \alpha_k = \frac{1}{\pi}\int_0^{2\pi}\phi(x)\cos kx\,dx,$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} F(x)\sin kx\,dx, \qquad \beta_k = \frac{1}{\pi}\int_0^{2\pi}\phi(x)\sin kx\,dx,$$

$$k = 0,\ 1,\ 2,\ \cdots.$$

Let
$$K = K(2\pi) = \int_0^{2\pi} F(x)\phi(x)\,dx,$$

$$K_n(x) = \frac{a_0}{2}\int_0^x F(x)\,dx + \sum_{k=1}^{n}\left(\alpha_k\int_0^x F(x)\cos kx\,dx + \beta_k\int_0^x F(x)\sin kx\,dx\right)$$

$$= \int_0^x F(x)\left\{\frac{a_0}{2} + \sum_{k=1}^{n}(\alpha_k\cos kx + \beta_k\sin kx)\right\}dx,$$

$$L_n(x) = \frac{a_0 x}{2} + \sum_{k=1}^{n}\left\{\frac{a_k}{k}\sin kx + \frac{\beta_k}{k}(1-\cos kx)\right\}.$$

It has been shown without the use of the second theorem of the mean* that

$$K = \pi\left[\frac{a_0 a_0}{2} + \sum_{k=1}^{\infty}(a_k\alpha_k + b_k\beta_k)\right],$$

$$L(x) = \frac{a_0 x}{2} + \sum_{k=1}^{\infty}\left\{\frac{a_k}{k}\sin kx + \frac{\beta_k}{k}(1-\cos kx)\right\}.$$

Moreover, the first of these series converges absolutely and the second converges absolutely and uniformly to the continuous function $L(x)$ in the interval $0 \leqq x \leqq 2\pi$. Hence

$$\lim_{n=\infty} K_n(2\pi) = K,$$

$$\lim_{n=\infty} L_n(x) = L(x), \qquad 0 \leqq x \leqq 2\pi.$$

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[*] Cf. de la Vallée-Poussin, loc. cit., p. 32.
Hurwitz, loc. cit., p. 438.
Bôcher, ANNALS OF MATHEMATICS, ser. 2, vol. 7 (1908), p. 107.

Now
$$L_n'(x) \equiv \frac{dL_n(x)}{dx} = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + \beta_k \sin kx,$$
and
$$L_n(x) = \int_0^x L_n'(x)\, dx, \qquad K_n(x) = \int_0^x F(x)\, L_n'(x)\, dx.$$

$L_n'(x)$ is a continuous function with but a finite number of changes of sign in the interval $0 \leqq x \leqq 2\pi$. To it, therefore, Netto's proof applies directly, and we have

$$K_n(2\pi) = L_n(\xi_n), \qquad 0 \leqq \xi_n \leqq 2\pi.$$

Hence
$$K = \lim_{n=\infty} K_n(2\pi) = \lim_{n=\infty} L_n(\xi_n).$$

Now the points ξ_n have at least one limiting point ξ. Let $\xi_{n_1}, \xi_{n_2}, \cdots \xi_{n_k}, \cdots$ be a sub-set of the set of points ξ_n converging to ξ as a limit. Then

$$\lim_{k=\infty} L_{n_k}(\xi_{n_k}) = \lim_{n=\infty} L_n(\xi_n),$$

and, because of the uniform convergence of $L_n(x)$ to the limiting function $L(x)$,

$$\lim_{k=\infty} L_{n_k}(\xi_{n_k}) = L(\xi).$$

Hence
$$L(\xi) = K, \qquad 0 \leqq \xi \leqq 2\pi;$$

in other words, there exists a ξ such that

$$\int_0^{2\pi} F(x)\,\phi(x)\,dx = \int_0^{\xi} \phi(x)\,dx, \qquad 0 \leqq \xi \leqq 2\pi,$$

from which we have at once the second theorem of the mean as stated in paragraph 2.

6. Application of the fundamental theorem of Fourier's constants to the proof by Abel's lemma. The method of paragraph 5 was there applied to complete the proof given by Netto. It is equally applicable to the proof by means of Abel's lemma. $L_n'(x)$ changes sign at most a finite number $m_n - 1$ of times in the interval $0 \leqq x \leqq 2\pi$. Let the points at which it changes sign be $x_2, x_3, \cdots x_{m_n}$, and let $0 < x_1 < x_2$. Then by the first theorem of the mean

$$\begin{aligned}
&K_n(x_1) - K_n(0) = \lambda_1 \{L_n(x_1) - L_n(0)\},\\
&K_n(x_2) - K_n(x_1) = \lambda_2 \{L_n(x_2) - L_n(x_1)\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&K_n(x_{m_n}) - K_n(x_{m_n-1}) = \lambda_{m_n}\{L_n(x_{m_n}) - L_n(x_{m_n-1})\},\\
&K_n(2\pi) - K_n(x_{m_n}) = \lambda_{m_n+1} \{L_n(2\pi) - L_n(x_{m_n})\};
\end{aligned}$$

where

$$1 \geqq \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_{m_n} \geqq \lambda_{m_n+1} \geqq 0,$$

and these λ's are numbers lying between the bounds of indetermination of $F(x)$ in the respective intervals $0 \leqq x \leqq x_1$, $x_1 \leqq x \leqq x_2$, $\cdots x_{m_n-1} \leqq x \leqq x_{m_n}$, $x_{m_n} \leqq x \leqq 2\pi$. Since $L_n(0) = K_n(0) = 0$ we have

$$\begin{aligned}
K_n(2\pi) = (\lambda_1 - \lambda_2)\, L_n(x_1) + (\lambda_2 - \lambda_3)\, L_n(x_2) \cdots\\
+ (\lambda_{m_n} - \lambda_{m_n+1})\, L_n(x_{m_n}) + \lambda_{m_n+1}\, L_n(2\pi).
\end{aligned}$$

Let l_n, L_n be respectively the minimum and the maximum of the continuous function $L_n(x)$ in $0 \leqq x \leqq 2\pi$. Then, since $\lambda_i - \lambda_{i+1} \geqq 0$, $\lambda_{m_n+1} \geqq 0$,

$$\begin{aligned}
l_n \cdot (\lambda_1 - \lambda_2 + \lambda_2 - \lambda_3 + \cdots + \lambda_{m_n} - \lambda_{m_n+1} + \lambda_{m_n+1}) \leqq K_n(2\pi)\\
\leqq L_n \cdot (\lambda_1 - \lambda_2 + \lambda_2 - \lambda_3 + \cdots + \lambda_{m_n} - \lambda_{m_n+1} + \lambda_{m_n+1}).
\end{aligned}$$

Hence

$$l_n \lambda_1 \leqq K_n(2\pi) \leqq \lambda_1 L_n.$$

Let now x_1 approach zero. Then

$$\lim_{x_1=0} \lambda_1 = 1.$$

Hence

$$l_n \leqq K_n(2\pi) \leqq L_n,$$

and therefore there exists at least one value ξ_n, $0 \leqq \xi_n \leqq 2\pi$, such that for $x = \xi_n$ the continuous function $L_n(x)$ takes on the value $K_n(2\pi)$, i. e.,

$$K_n(2\pi) = L_n(\xi_n).$$

The proof from this point onward then proceeds as in paragraph 5.

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